

A semigroup approach to Finsler geometry: Bakry–Ledoux’s isoperimetric inequality

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Abstract

We develop the celebrated semigroup approach à la Bakry et al on Finsler manifolds, where natural Laplacian and heat semigroup are nonlinear, based on the Bochner–Weitzenböck formula established by Sturm and the author. We show the L^2 - and L^1 -gradient estimates on (possibly non-compact) Finsler manifolds under a mild uniform smoothness assumption. These estimates are equivalent to a lower weighted Ricci curvature bound and the Bochner inequalities. As a geometric application, we prove Bakry–Ledoux’s Gaussian isoperimetric inequality in the sharp form. This extends Cavalletti–Mondino’s inequality on reversible Finsler manifolds to non-reversible spaces, and also improves the author’s recent estimate, both based on the localization method.

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1 Introduction

The aim of this article is to put forward the semigroup approach in geometric analysis on Finsler manifolds, based on the Bochner–Weitzenböck formula established in [OS3]. There are already a number of applications of the Bochner–Weitzenböck formula (including [WX, Xi, YH, Oh5]). We believe that the machinery in this article will further accelerate the development of geometric analysis on Finsler manifolds. In addition, our treatment of a nonlinear generator and the associated nonlinear semigroup (Laplacian and heat semigroup) would be of independent interest from the analytic viewpoint.

The celebrated theory developed by Bakry, Émery, Ledoux et al (sometimes called the Γ -calculus) studies symmetric generators and the associated linear, symmetric diffusion semigroups under a kind of Bochner inequality (called the *(analytic) curvature-dimension condition*). Attributed to Bakry–Émery’s original work [BE], this condition will be denoted by $\text{BE}(K, N)$ in this introduction, where $K \in \mathbb{R}$ and $N \in (1, \infty]$ are parameters corresponding to ‘curvature’ and ‘dimension’. This technique is extremely powerful in studying various inequalities (log-Sobolev and Poincaré inequalities, gradient estimates, etc.) in a unified way, we refer to [BE, Ba] and the recent book [BGL] for more on this theory.

On a Riemannian manifold equipped with the Laplacian Δ , $\text{BE}(K, N)$ means the following Bochner-type inequality:

$$\Delta \left[\frac{\|\nabla u\|^2}{2} \right] - \langle \nabla(\Delta u), \nabla u \rangle \geq K \|\nabla u\|^2 + \frac{(\Delta u)^2}{N}.$$

Thus a Riemannian manifold with Ricci curvature not less K and dimension not greater than N (more generally, a weighted Riemannian manifold of weighted Ricci curvature $\text{Ric}_N \geq K$) is a fundamental example satisfying $\text{BE}(K, N)$.

Later, inspired by [CMS, OV], Sturm [vRS, St1, St2] and Lott–Villani [LV] introduced the *(geometric) curvature-dimension condition* $\text{CD}(K, N)$ for metric measure spaces in terms of optimal transport theory. The condition $\text{CD}(K, N)$ characterizes $\text{Ric} \geq K$ and $\dim \leq N$ (or $\text{Ric}_N \geq K$) for (weighted) Riemannian manifolds, and its formulation requires a lower regularity of spaces than $\text{BE}(K, N)$. We refer to Villani’s book [Vi] for more on this rapidly developing theory. It was shown in [Oh2] that $\text{CD}(K, N)$ also holds and characterizes $\text{Ric}_N \geq K$ for Finsler manifolds, where the natural Laplacian and the associated heat semigroup are nonlinear. For this reason, Ambrosio, Gigli and Savaré [AGS1] introduced a reinforced version $\text{RCD}(K, \infty)$ called the *Riemannian curvature-dimension condition* as the combination of $\text{CD}(K, \infty)$ and the linearity of heat semigroup, followed by the finite-dimensional analogue $\text{RCD}^*(K, N)$ investigated by Erbar, Kuwada and Sturm [EKS]. It then turned out that $\text{RCD}^*(K, N)$ is equivalent to $\text{BE}(K, N)$ ([AGS2, EKS]); this equivalence justifies the term ‘curvature-dimension condition’ which actually came from the similarity to Bakry’s theory.

In this article, we develop the former theory of Bakry et al on Finsler manifolds. We consider a Finsler manifold M equipped with a Finsler metric $F : TM \rightarrow [0, \infty)$ and a positive \mathcal{C}^∞ -measure \mathfrak{m} on M . We will not assume that F is *reversible*, thus $F(-v) \neq F(v)$ is allowed. The key ingredient, the *Bochner inequality* under $\text{Ric}_N \geq K$, was established

in [OS3] as follows:

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) \geq KF^2(\nabla u) + \frac{(\Delta u)^2}{N}. \quad (1.1)$$

This Bochner inequality has the same form as the Riemannian case by means of a mixture of the nonlinear Laplacian Δ and its linearization $\Delta^{\nabla u}$. Despite of this mixture, we could derive Bakry–Émery’s L^2 -gradient estimate as well as Li–Yau’s estimates on compact manifolds (see [OS3, §4]). We proceed in this way and show the *improved Bochner inequality* under $\text{Ric}_\infty \geq K$ (Proposition 3.5):

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) - KF^2(\nabla u) \geq D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)]). \quad (1.2)$$

The first application of (1.2) is the L^1 -gradient estimate (Theorem 3.7). We also see that the Bochner inequalities (1.1), (1.2) and the L^2 - and L^1 -gradient estimates are all equivalent to $\text{Ric}_\infty \geq K$ (Theorem 3.8), under the mild uniform smoothness assumption $S_F < \infty$ (see below).

The second and geometric application of (1.2) is a generalization of *Bakry–Ledoux’s Gaussian isoperimetric inequality* (Theorem 4.1):

Theorem (Bakry–Ledoux’s isoperimetric inequality) *Let (M, F, \mathbf{m}) be complete and satisfy $\text{Ric}_\infty \geq K > 0$, $\mathbf{m}(M) = 1$ and $S_F < \infty$. Then we have*

$$\mathcal{I}_{(M, F, \mathbf{m})}(\theta) \geq \mathcal{I}_K(\theta) \quad (1.3)$$

for all $\theta \in [0, 1]$, where

$$\mathcal{I}_K(\theta) := \sqrt{\frac{K}{2\pi}} e^{-Kc^2(\theta)/2} \quad \text{with} \quad \theta = \int_{-\infty}^{c(\theta)} \sqrt{\frac{K}{2\pi}} e^{-Ka^2/2} da.$$

Here $\mathcal{I}_{(M, F, \mathbf{m})} : [0, 1] \rightarrow [0, \infty)$ is the *isoperimetric profile* defined as the least boundary area of sets $A \subset M$ with $\mathbf{m}(A) = \theta$ (see the beginning of Section 4 for the precise definition), and

$$S_F := \sup_{x \in M} \sup_{v, w \in T_x M \setminus 0} \frac{g_v(w, w)}{F^2(w)} \in [1, \infty]$$

is the (2-)uniform smoothness constant which also bounds the *reversibility*:

$$\Lambda_F := \sup_{v \in TM \setminus 0} \frac{F(v)}{F(-v)} \in [1, \infty] \quad (1.4)$$

as $\Lambda_F \leq \sqrt{S_F}$ (see Lemma 2.4). (In particular, the forward completeness is equivalent to the backward completeness, we denoted it by the plain *completeness* in the theorem.) The condition $S_F < \infty$ will appear several times in our argument (Theorem 2.8, Propositions 3.1, 4.3), while the uniform convexity does not play any role. We do not know whether this is merely a coincidence.

The inequality (1.3) has the same form as the Riemannian case in [BL], thus it is sharp and a model space is the real line \mathbb{R} equipped with the normal (Gaussian) distribution $d\mathbf{m} = \sqrt{K/2\pi} e^{-Kx^2/2} dx$. See [BL] for the original work of Bakry and Ledoux on general linear diffusion semigroups (influenced by Bobkov's works [Bob1, Bob2]), and [Bor, SC] for the classical Euclidean or Hilbert cases.

The above theorem (1.3) extends Cavalletti–Mondino's isoperimetric inequality in [CM] to non-reversible Finsler manifolds. Precisely, in [CM] they considered essentially non-branching metric measure spaces (X, d, \mathbf{m}) satisfying $\text{CD}(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty)$, and showed the sharp *Lévy–Gromov type isoperimetric inequality* of the form

$$\mathcal{I}_{(X, d, \mathbf{m})}(\theta) \geq \mathcal{I}_{K, N, D}(\theta)$$

with $\text{diam } X \leq D (\leq \infty)$. The case of $N = \infty$ is not included in [CM] for technical reasons on the structure of $\text{CD}(K, N)$ -spaces, but the same argument gives (1.3) (corresponding to $N = D = \infty$) for reversible Finsler manifolds. The proof in [CM] is based on the *localization method* (also called a *needle decomposition*) inspired by Klartag's work [Kl] on Riemannian manifolds, extending the successful technique in convex geometry going back to [PW, GM, LS, KLS]. Along the lines of [CM], in [Oh7] we have generalized the localization method to non-reversible Finsler manifolds, however, then we obtain only a weaker isoperimetric inequality:

$$\mathcal{I}_{(M, F, \mathbf{m})}(\theta) \geq \Lambda_F^{-1} \cdot \mathcal{I}_{K, N, D}(\theta), \quad (1.5)$$

where Λ_F is the reversibility constant as in (1.4). The inequality (1.3) improves (1.5) in the case where $N = D = \infty$ and $K > 0$, and supports a conjecture that the sharp isoperimetric inequality in the non-reversible case is the same as the reversible case, namely Λ_F^{-1} in (1.5) would be removed.

The organization of this article is as follows: In Section 2 we review the basics of Finsler geometry, including the weighted Ricci curvature and the Bochner–Weitzenböck formula. We slightly generalize the integrated form of the Bochner inequality in [OS3] to fit in our setting. Section 3 is devoted to a detailed study of the nonlinear heat semigroup and its linearizations. We improve the Bochner inequality under $\text{Ric}_\infty \geq K$ and show the L^1 -gradient estimate. Finally, we prove Bakry–Ledoux's Gaussian isoperimetric inequality in Section 4.

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2 Geometry and analysis on Finsler manifolds

We review the basics of Finsler geometry (we refer to [BCS, Sh] for further reading), and introduce the weighted Ricci curvature and the nonlinear Laplacian studied in [Oh2, OS1] (see also [GS] for the latter).

Throughout the article, let M be a connected, n -dimensional \mathcal{C}^∞ -manifold without boundary such that $n \geq 2$. We also fix an arbitrary positive \mathcal{C}^∞ -measure \mathbf{m} on M .

2.1 Finsler manifolds

Given a local coordinate $(x^i)_{i=1}^n$ on an open set $U \subset M$, we will always use the fiber-wise linear coordinate $(x^i, v^j)_{i,j=1}^n$ of TU such that

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j} \Big|_x \in T_x M, \quad x \in U.$$

Definition 2.1 (Finsler structures) We say that a nonnegative function $F : TM \rightarrow [0, \infty)$ is a C^∞ -Finsler structure of M if the following three conditions hold:

- (1) (*Regularity*) F is C^∞ on $TM \setminus 0$, where 0 stands for the zero section;
- (2) (*Positive 1-homogeneity*) It holds $F(cv) = cF(v)$ for all $v \in TM$ and $c \geq 0$;
- (3) (*Strong convexity*) The $n \times n$ matrix

$$(g_{ij}(v))_{i,j=1}^n := \left(\frac{1}{2} \frac{\partial^2 (F^2)}{\partial v^i \partial v^j} (v) \right)_{i,j=1}^n \quad (2.1)$$

is positive-definite for all $v \in TM \setminus 0$.

We call such a pair (M, F) a C^∞ -Finsler manifold.

In other words, F provides a Minkowski norm on each tangent space which varies smoothly in horizontal directions. If $F(-v) = F(v)$ holds for all $v \in TM$, then we say that F is *reversible* or *absolutely homogeneous*. The strong convexity means that the unit sphere $T_x M \cap F^{-1}(1)$ (called the *indicatrix*) is ‘positively curved’ and implies the strict convexity: $F(v+w) \leq F(v) + F(w)$ for all $v, w \in T_x M$ and equality holds only when $v = aw$ or $w = av$ for some $a \geq 0$.

In the coordinate $(x^i, \alpha_j)_{i,j=1}^n$ of T^*U given by $\alpha = \sum_{j=1}^n \alpha_j dx^j$, we will also consider

$$g_{ij}^*(\alpha) := \frac{1}{2} \frac{\partial^2 [(F^*)^2]}{\partial \alpha_i \partial \alpha_j}(\alpha), \quad i, j = 1, 2, \dots, n,$$

for $\alpha \in T^*U \setminus 0$. Here $F^* : T^*M \rightarrow [0, \infty)$ is the *dual Minkowski norm* to F , namely

$$F^*(\alpha) := \sup_{v \in T_x M, F(v) \leq 1} \alpha(v) = \sup_{v \in T_x M, F(v)=1} \alpha(v)$$

for $\alpha \in T_x^* M$. It is clear by definition that $\alpha(v) \leq F^*(\alpha)F(v)$, and hence

$$\alpha(v) \geq -F^*(\alpha)F(-v), \quad \alpha(v) \geq -F^*(-\alpha)F(v).$$

We remark (and stress) that, however, $\alpha(v) \geq -F^*(\alpha)F(v)$ does not hold in general.

Let us denote by $\mathcal{L}^* : T^*M \rightarrow TM$ the *Legendre transform*. Precisely, \mathcal{L}^* is sending $\alpha \in T_x^* M$ to the unique element $v \in T_x M$ such that $F(v) = F^*(\alpha)$ and $\alpha(v) = F^*(\alpha)^2$. In coordinates we can write down

$$\mathcal{L}^*(\alpha) = \sum_{i,j=1}^n g_{ij}^*(\alpha) \alpha_i \frac{\partial}{\partial x^j} \Big|_x = \sum_{j=1}^n \frac{1}{2} \frac{\partial [(F^*)^2]}{\partial \alpha_j}(\alpha) \frac{\partial}{\partial x^j} \Big|_x$$

for $\alpha \in T_x^*M \setminus 0$ (the latter expression makes sense also at 0). Note that

$$g_{ij}^*(\alpha) = g^{ij}(\mathcal{L}^*(\alpha)) \quad \text{for } \alpha \in T_x^*M \setminus 0,$$

where $(g^{ij}(v))$ denotes the inverse matrix of $(g_{ij}(v))$. The map $\mathcal{L}^*|_{T_x^*M}$ is being a linear operator only when $F|_{T_xM}$ comes from an inner product. We also define $\mathcal{L} := (\mathcal{L}^*)^{-1} : TM \rightarrow T^*M$.

For $x, y \in M$, we define the (nonsymmetric) *distance* from x to y by

$$d(x, y) := \inf_{\eta} \int_0^1 F(\dot{\eta}(t)) dt,$$

where the infimum is taken over all \mathcal{C}^1 -curves $\eta : [0, 1] \rightarrow M$ such that $\eta(0) = x$ and $\eta(1) = y$. Note that $d(y, x) \neq d(x, y)$ can happen since F is only positively homogeneous. A \mathcal{C}^∞ -curve η on M is called a *geodesic* if it is locally minimizing and has a constant speed with respect to d , similarly to Riemannian or metric geometry. See (2.7) below for the precise geodesic equation. For $v \in T_xM$, if there is a geodesic $\eta : [0, 1] \rightarrow M$ with $\dot{\eta}(0) = v$, then we define the *exponential map* by $\exp_x(v) := \eta(1)$. We say that (M, F) is *forward complete* if the exponential map is defined on whole TM . Then the Hopf–Rinow theorem ensures that any pair of points is connected by a minimal geodesic (see [BCS, Theorem 6.6.1]).

Given each $v \in T_xM \setminus 0$, the positive-definite matrix $(g_{ij}(v))_{i,j=1}^n$ in (2.1) induces the Riemannian structure g_v of T_xM via

$$g_v \left(\sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_x, \sum_{j=1}^n b_j \frac{\partial}{\partial x^j} \Big|_x \right) := \sum_{i,j=1}^n g_{ij}(v) a_i b_j. \quad (2.2)$$

Notice that this definition is coordinate-free, and we have $g_v(v, v) = F^2(v)$. One can regard g_v as the best Riemannian approximation of $F|_{T_xM}$ in the direction v . In fact, the unit sphere of g_v is tangent to that of $F|_{T_xM}$ at $v/F(v)$ up to the second order. The *Cartan tensor*

$$A_{ijk}(v) := \frac{F(v)}{2} \frac{\partial g_{ij}}{\partial v^k}(v), \quad v \in TM \setminus 0,$$

measures the variation of g_v in the vertical directions, and vanishes everywhere on $TM \setminus 0$ if and only if F comes from a Riemannian metric.

The following useful fact on homogeneous functions (see [BCS, Theorem 1.2.1]) plays fundamental roles in our calculus.

Theorem 2.2 (Euler’s theorem) *Suppose that a differentiable function $H : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}$ satisfies $H(cv) = c^r H(v)$ for some $r \in \mathbb{R}$ and all $c > 0$ and $v \in \mathbb{R}^n \setminus 0$ (that is, positively r -homogeneous). Then we have*

$$\sum_{i=1}^n \frac{\partial H}{\partial v^i}(v) v^i = r H(v)$$

for all $v \in \mathbb{R}^n \setminus 0$.

Observe that g_{ij} is positively 0-homogeneous on each $T_x M$, and hence

$$\sum_{i=1}^n A_{ijk}(v)v^i = \sum_{j=1}^n A_{ijk}(v)v^j = \sum_{k=1}^n A_{ijk}(v)v^k = 0 \quad (2.3)$$

for all $v \in TM \setminus 0$ and $i, j, k = 1, 2, \dots, n$. Define the *formal Christoffel symbol*

$$\gamma_{jk}^i(v) := \frac{1}{2} \sum_{l=1}^n g^{il}(v) \left\{ \frac{\partial g_{lk}}{\partial x^j}(v) + \frac{\partial g_{jl}}{\partial x^k}(v) - \frac{\partial g_{jk}}{\partial x^l}(v) \right\} \quad (2.4)$$

for $v \in TM \setminus 0$. We also introduce the *geodesic spray coefficients* and the *nonlinear connection*

$$G^i(v) := \sum_{j,k=1}^n \gamma_{jk}^i(v)v^j v^k, \quad N_j^i(v) := \frac{1}{2} \frac{\partial G^i}{\partial v^j}(v)$$

for $v \in TM \setminus 0$, and $G^i(0) = N_j^i(0) := 0$ by convention. Note that G^i is positively 2-homogeneous, hence Theorem 2.2 implies $\sum_{j=1}^n N_j^i(v)v^j = G^i(v)$.

By using the nonlinear connections N_j^i , the coefficients of the *Chern connection* are given by

$$\Gamma_{jk}^i(v) := \gamma_{jk}^i(v) - \sum_{l,m=1}^n \frac{g^{il}}{F} (A_{lkm} N_j^m + A_{jlm} N_k^m - A_{jkm} N_l^m)(v) \quad (2.5)$$

on $TM \setminus 0$. The corresponding *covariant derivative* of a vector field X by $v \in T_x M$ with *reference vector* $w \in T_x M \setminus 0$ is defined as

$$D_v^w X(x) := \sum_{i,j=1}^n \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \sum_{k=1}^n \Gamma_{jk}^i(w) v^j X^k(x) \right\} \frac{\partial}{\partial x^i} \Big|_x \in T_x M. \quad (2.6)$$

Then the *geodesic equation* is written as, with the help of (2.3),

$$D_{\dot{\eta}}^{\dot{\eta}} \dot{\eta}(t) = \sum_{i=1}^n \left\{ \ddot{\eta}^i(t) + G^i(\dot{\eta}(t)) \right\} \frac{\partial}{\partial x^i} \Big|_{\eta(t)} = 0. \quad (2.7)$$

2.2 Uniform smoothness

We will need the following quantity associated with (M, F) :

$$\mathbf{S}_F := \sup_{x \in M} \sup_{v, w \in T_x M \setminus 0} \frac{g_v(w, w)}{F^2(w)} \in [1, \infty].$$

Since $g_v(w, w) \leq \mathbf{S}_F F^2(w)$ and g_v is the Hessian of $F^2/2$ at v , the constant \mathbf{S}_F measures the (fiber-wise) concavity of F^2 and is called the *(2-)uniform smoothness constant* (see [Oh1]). We remark that $\mathbf{S}_F = 1$ holds if and only if (M, F) is Riemannian. The following lemma is a standard fact, we give a proof for thoroughness.

Lemma 2.3 For any $x \in M$ and $v \in T_x M \setminus 0$, we have

$$\sup_{w \in T_x M \setminus 0} \frac{g_v(w, w)}{F^2(w)} = \sup_{\beta \in T_x^* M \setminus 0} \frac{F^*(\beta)^2}{g_\alpha^*(\beta, \beta)},$$

where we put $\alpha := \mathcal{L}(v)$ and

$$g_\alpha^*(\beta, \beta) := \sum_{i,j=1}^n g_{ij}^*(\alpha) \beta_i \beta_j, \quad \beta = \sum_{i=1}^n \beta_i dx^i,$$

is an inner product of $T_x^* M$.

Proof. Choose a coordinate $(x^i)_{i=1}^n$ around x such that $g_{ij}(v) = \delta_{ij}$ and set

$$\mathbb{S}_x := \left\{ w = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i} \in T_x M \mid \sum_{i=1}^n (w^i)^2 = 1 \right\},$$

$$\mathbb{S}_x^* := \left\{ \beta = \sum_{i=1}^n \beta_i dx^i \in T_x^* M \mid \sum_{i=1}^n (\beta_i)^2 = 1 \right\}.$$

Given $w \in \mathbb{S}_x$, we first take $\beta \in \mathbb{S}_x^*$ such that $\beta(w) = 1$. Then we have $1 = \beta(w) \leq F^*(\beta)F(w)$ and hence

$$\frac{g_v(w, w)}{F^2(w)} = \frac{1}{F^2(w)} \leq F^*(\beta)^2 = \frac{F^*(\beta)^2}{g_\alpha^*(\beta, \beta)}.$$

Next we consider $\beta' \in \mathbb{S}_x^*$ with $\beta'(w) = F^*(\beta')F(w)$, then $F^*(\beta')F(w) = \beta'(w) \leq 1$ and hence $1/F^2(w) \geq F^*(\beta')^2$. This completes the proof. \square

Although it will not be used in the sequel, one can in a similar manner introduce the (2-)uniform convexity constant:

$$C_F := \sup_{x \in M} \sup_{v, w \in T_x M \setminus 0} \frac{F^2(w)}{g_v(w, w)} = \sup_{x \in M} \sup_{\alpha, \beta \in T_x^* M \setminus 0} \frac{g_\alpha^*(\beta, \beta)}{F^*(\beta)^2} \in [1, \infty].$$

Again, $C_F = 1$ holds if and only if (M, F) is Riemannian.

We remark that S_F and C_F control the *reversibility constant*, defined by

$$\Lambda_F := \sup_{v \in TM \setminus 0} \frac{F(v)}{F(-v)} \in [1, \infty],$$

as follows.

Lemma 2.4 We have

$$\Lambda_F \leq \min\{\sqrt{S_F}, \sqrt{C_F}\}.$$

Proof. For any $v \in TM \setminus 0$, we observe

$$\frac{F^2(v)}{F^2(-v)} = \frac{g_v(v, v)}{F^2(-v)} = \frac{g_v(-v, -v)}{F^2(-v)} \leq S_F,$$

and similarly

$$\frac{F^2(v)}{F^2(-v)} = \frac{F^2(v)}{g_{-v}(v, v)} \leq C_F.$$

\square

2.3 Weighted Ricci curvature

The *Ricci curvature* (as the trace of the *flag curvature*) on a Finsler manifold is defined by using some connection. Instead of giving a precise definition in coordinates (for which we refer to [BCS]), here we explain a useful interpretation in [Sh, §6.2]. Given a unit vector $v \in T_x M \cap F^{-1}(1)$, we extend it to a \mathcal{C}^∞ -vector field V on a neighborhood U of x in such a way that every integral curve of V is geodesic, and consider the Riemannian structure g_V of U induced from (2.2). Then the *Finsler Ricci curvature* $\text{Ric}(v)$ of v with respect to F coincides with the *Riemannian Ricci curvature* of v with respect to g_V (in particular, it is independent of the choice of V).

Inspired by the above interpretation of the Ricci curvature as well as the theory of weighted Ricci curvature (also called the *Bakry–Émery–Ricci curvature*) of Riemannian manifolds, the *weighted Ricci curvature* for (M, F, \mathbf{m}) was introduced in [Oh2] as follows (though our main concern is the case of $N = \infty$, we will deal with general N in this section for completeness). Recall that \mathbf{m} is a positive \mathcal{C}^∞ -measure on M , from here on it comes into play.

Definition 2.5 (Weighted Ricci curvature) Given a unit vector $v \in T_x M$, let V be a \mathcal{C}^∞ -vector field on a neighborhood U of x as above. We decompose \mathbf{m} as $\mathbf{m} = e^{-\Psi} \text{vol}_{g_V}$ on U , where $\Psi \in \mathcal{C}^\infty(U)$ and vol_{g_V} is the volume form of g_V . Denote by $\eta : (-\varepsilon, \varepsilon) \rightarrow M$ the geodesic such that $\dot{\eta}(0) = v$. Then, for $N \in (-\infty, 0) \cup (n, \infty)$, define

$$\text{Ric}_N(v) := \text{Ric}(v) + (\Psi \circ \eta)''(0) - \frac{(\Psi \circ \eta)'(0)^2}{N - n}.$$

We also define as the limits:

$$\text{Ric}_\infty(v) := \text{Ric}(v) + (\Psi \circ \eta)''(0), \quad \text{Ric}_n(v) := \lim_{N \downarrow n} \text{Ric}_N(v).$$

For $c \geq 0$, we set $\text{Ric}_N(cv) := c^2 \text{Ric}_N(v)$.

We will denote by $\text{Ric}_N \geq K$, $K \in \mathbb{R}$, the condition $\text{Ric}_N(v) \geq KF^2(v)$ for all $v \in TM$. In the Riemannian case, the study of Ric_∞ goes back to Lichnerowicz [Li], he showed a Cheeger–Gromoll type splitting theorem (see [FLZ, WW] for some generalizations, and [Oh5] for Finsler counterparts). The range $N \in (n, \infty)$ has been well studied by Bakry [Ba], Qian [Qi] and many others. The study of the range $N \in (-\infty, 0)$ is more recent; see [Mi2] for isoperimetric inequalities, [Oh6] for the curvature-dimension condition, and [Wy] for splitting theorems (for $N \in (-\infty, 1]$). Some historic accounts on related works concerning $N < 0$ in convex geometry and partial differential equations can be found in [Mi2, Mi3].

It is established in [Oh2] (and [Oh6] for $N < 0$, [Oh7] for $N = 0$) that, for $K \in \mathbb{R}$, the bound $\text{Ric}_N \geq K$ is equivalent to Lott, Sturm and Villani’s *curvature-dimension condition* $\text{CD}(K, N)$. This extends the corresponding result on weighted Riemannian manifolds and has many geometric and analytic applications (see [Oh2, OS1] among others).

Remark 2.6 (S-curvature) For a Riemannian manifold (M, g, vol_g) endowed with the Riemannian volume measure, clearly we have $\Psi \equiv 0$ and hence $\text{Ric}_N = \text{Ric}$ for all N .

It is also known that, for Finsler manifolds of *Berwald type* (i.e., Γ_{ij}^k is constant on each $T_x M \setminus 0$), the *Busemann–Hausdorff measure* satisfies $(\Psi \circ \eta)' \equiv 0$ (in other words, Shen’s **S**-curvature vanishes, see [Sh, §7.3]). In general, however, there may not exist any measure with vanishing **S**-curvature (see [Oh3] for such an example). This is a reason why we chose to begin with an arbitrary measure \mathbf{m} .

2.4 Nonlinear Laplacian and heat flow

For a differentiable function $u : M \rightarrow \mathbb{R}$, the *gradient vector* at x is defined as the Legendre transform of the derivative of u : $\nabla u(x) := \mathcal{L}^*(Du(x)) \in T_x M$. If $Du(x) \neq 0$, then we can write down in coordinates as

$$\nabla u = \sum_{i,j=1}^n g_{ij}^*(Du) \frac{\partial u}{\partial x^j} \frac{\partial}{\partial x^i}.$$

We need to be careful when $Du(x) = 0$, because $g_{ij}^*(Du(x))$ is not defined as well as the Legendre transform \mathcal{L}^* is only continuous at the zero section. Thus we set

$$M_u := \{x \in M \mid Du(x) \neq 0\}.$$

For a twice differentiable function $u : M \rightarrow \mathbb{R}$ and $x \in M_u$, we define a kind of *Hessian* $\nabla^2 u(x) \in T_x^* M \otimes T_x M$ by using the covariant derivative (2.6) as

$$\nabla^2 u(v) := D_v^{\nabla u}(\nabla u)(x) \in T_x M, \quad v \in T_x M.$$

The operator $\nabla^2 u(x)$ is symmetric in the sense that

$$g_{\nabla u}(\nabla^2 u(v), w) = g_{\nabla u}(v, \nabla^2 u(w))$$

for all $v, w \in T_x M$ with $x \in M_u$ (see, for example, [OS3, Lemma 2.3]).

Define the *divergence* of a differentiable vector field V on M with respect to the measure \mathbf{m} by

$$\operatorname{div}_{\mathbf{m}} V := \sum_{i=1}^n \left(\frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i} \right), \quad V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i},$$

where we decomposed \mathbf{m} as $d\mathbf{m} = e^\Phi dx^1 dx^2 \cdots dx^n$. One can rewrite in the weak form as

$$\int_M \phi \operatorname{div}_{\mathbf{m}} V d\mathbf{m} = - \int_M D\phi(V) d\mathbf{m} \quad \text{for all } \phi \in \mathcal{C}_c^\infty(M),$$

that makes sense for measurable vector fields V with $F(V) \in L_{\text{loc}}^1(M)$. Then we define the distributional *Laplacian* of $u \in H_{\text{loc}}^1(M)$ by $\Delta u := \operatorname{div}_{\mathbf{m}}(\nabla u)$ in the weak sense that

$$\int_M \phi \Delta u d\mathbf{m} := - \int_M D\phi(\nabla u) d\mathbf{m} \quad \text{for all } \phi \in \mathcal{C}_c^\infty(M).$$

Notice that $H_{\text{loc}}^1(M)$ is defined solely in terms of the differentiable structure of M . Since taking the gradient vector (more precisely, the Legendre transform) is a nonlinear operation, our Laplacian Δ is a nonlinear operator unless F is Riemannian.

In [OS1, OS3], we have studied the associated *nonlinear heat equation* $\partial_t u = \Delta u$. To recall some results in [OS1], we define the *Dirichlet energy* of $u \in H_{\text{loc}}^1(M)$ by

$$\mathcal{E}(u) := \frac{1}{2} \int_M F^2(\nabla u) d\mathbf{m} = \frac{1}{2} \int_M F^*(Du)^2 d\mathbf{m}.$$

We remark that $\mathcal{E}(u) < \infty$ does not necessarily imply $\mathcal{E}(-u) < \infty$. Define $H_0^1(M)$ as the closure of $\mathcal{C}_c^\infty(M)$ with respect to the (absolutely homogeneous) norm

$$\|u\|_{H^1} := \|u\|_{L^2} + \{\mathcal{E}(u) + \mathcal{E}(-u)\}^{1/2}.$$

Note that $(H_0^1(M), \|\cdot\|_{H^1})$ is a Banach space.

Definition 2.7 (Global solutions) We say that a function u on $[0, T] \times M$, $T > 0$, is a *global solution* to the heat equation $\partial_t u = \Delta u$ if it satisfies the following:

- (1) $u \in L^2([0, T], H_0^1(M)) \cap H^1([0, T], H^{-1}(M))$;
- (2) We have

$$\int_M \phi \cdot \partial_t u_t d\mathbf{m} = - \int_M D\phi(\nabla u_t) d\mathbf{m}$$

for all $t \in [0, T]$ and $\phi \in \mathcal{C}_c^\infty(M)$, where we set $u_t := u(t, \cdot)$.

We refer to [Ev] for notations as in (1). Denoted by $H^{-1}(M)$ is the dual Banach space of $H_0^1(M)$. By noticing

$$\begin{aligned} \int_M |(D\phi - D\bar{\phi})(\nabla u_t)| d\mathbf{m} &\leq \int_M \max\{F^*(D[\phi - \bar{\phi}]), F^*(D[\bar{\phi} - \phi])\} F(\nabla u_t) d\mathbf{m} \\ &\leq \{2\mathcal{E}(\phi - \bar{\phi}) + 2\mathcal{E}(\bar{\phi} - \phi)\}^{1/2} \cdot \{2\mathcal{E}(u_t)\}^{1/2}, \end{aligned}$$

the test function ϕ can be taken from $H_0^1(M)$. Global solutions are constructed as gradient curves of the energy functional \mathcal{E} in the Hilbert space $L^2(M)$. We summarize the existence and regularity properties established in [OS1, §§3, 4] in the next theorem.

Theorem 2.8 (i) *For each initial datum $u_0 \in H_0^1(M)$ and $T > 0$, there exists a unique global solution u to the heat equation on $[0, T] \times M$, and the distributional Laplacian Δu_t is absolutely continuous with respect to \mathbf{m} for all $t \in (0, T)$.*

- (ii) *One can take the continuous version of a global solution u , and it enjoys the H_{loc}^2 -regularity in x as well as the $\mathcal{C}^{1,\alpha}$ -regularity in both t and x . Moreover, $\partial_t u$ lies in $H_{\text{loc}}^1(M) \cap \mathcal{C}(M)$, and further in $H_0^1(M)$ if $S_F < \infty$.*

We remark that the usual elliptic regularity yields that u is \mathcal{C}^∞ on

$$\bigcup_{t>0} (\{t\} \times M_{u_t}) = \{(t, x) \in (0, \infty) \times M \mid Du_t(x) \neq 0\}.$$

The proof of $\partial_t u \in H_0^1(M)$ under $S_F < \infty$ can be found in [OS1, Appendix A]. The uniqueness in (i) is a consequence of the following estimate similar to [OS1, Proposition 3.5].

Lemma 2.9 (Non-expansion in L^2) *For any global solutions u, \bar{u} to the heat equation,*

$$\partial_t(\|u_t - \bar{u}_t\|_{L^2}) \leq 0$$

holds for all $t > 0$. In particular, if $u_0 = \bar{u}_0$ almost everywhere, then $u_t = \bar{u}_t$ almost everywhere for all $t > 0$.

Proof. By employing $u_t - \bar{u}_t$ as a test function, we find

$$\begin{aligned} \partial_t(\|u_t - \bar{u}_t\|_{L^2}) &= \int_M 2(u_t - \bar{u}_t)(\partial_t u_t - \partial_t \bar{u}_t) d\mathbf{m} \\ &= -2 \int_M D[u_t - \bar{u}_t](\nabla u_t - \nabla \bar{u}_t) d\mathbf{m}. \end{aligned}$$

The convexity of $(F^*)^2$ yields that

$$D[u_t - \bar{u}_t](\nabla u_t - \nabla \bar{u}_t) = (Du_t - D\bar{u}_t)(\mathcal{L}^*(Du_t) - \mathcal{L}^*(D\bar{u}_t)) \geq 0,$$

which shows the claim. \square

We finally remark that, by the construction of heat flow as the gradient flow of \mathcal{E} , it is readily seen that:

$$\text{If } u_0 \geq 0 \text{ almost everywhere, then } u_t \geq 0 \text{ almost everywhere for all } t > 0. \quad (2.8)$$

Indeed, if $u_t < 0$ on a non-null set, then the curve $\bar{u}_t := \max\{u_t, 0\}$ will give a less energy with a less L^2 -length, a contradiction.

2.5 Bochner–Weitzenböck formula

Given $f \in H_{\text{loc}}^1(M)$ and a measurable vector field V such that $V \neq 0$ almost everywhere on M_f , we can define the gradient vector and the Laplacian on the weighted Riemannian manifold (M, g_V, \mathbf{m}) by

$$\nabla^V f := \begin{cases} \sum_{i,j=1}^n g^{ij}(V) \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i} & \text{on } M_f, \\ 0 & \text{on } M \setminus M_f, \end{cases} \quad \Delta^V f := \text{div}_{\mathbf{m}}(\nabla^V f),$$

where the latter is in the sense of distributions. We have $\nabla^{\nabla^u} u = \nabla u$ and $\Delta^{\nabla^u} u = \Delta u$ for $u \in H_{\text{loc}}^1(M)$ ([OS1, Lemma 2.4]). We also observe that, given $u, f_1, f_2 \in H_{\text{loc}}^1(M)$,

$$Df_2(\nabla^{\nabla^u} f_1) = g_{\nabla^u}(\nabla^{\nabla^u} f_1, \nabla^{\nabla^u} f_2) = Df_1(\nabla^{\nabla^u} f_2). \quad (2.9)$$

Theorem 2.10 (Bochner–Weitzenböck formula) *Given $u \in \mathcal{C}^\infty(M)$, we have*

$$\Delta^{\nabla^u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) = \text{Ric}_\infty(\nabla u) + \|\nabla^2 u\|_{\text{HS}(\nabla u)}^2 \quad (2.10)$$

as well as

$$\Delta^{\nabla^u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) \geq \text{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}$$

for $N \in (-\infty, 0) \cup [n, \infty]$ point-wise on M_u , where $\|\cdot\|_{\text{HS}(\nabla u)}$ stands for the Hilbert–Schmidt norm with respect to $g_{\nabla u}$.

In particular, if $\text{Ric}_N \geq K$, then we have

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) \geq KF^2(\nabla u) + \frac{(\Delta u)^2}{N} \quad (2.11)$$

on M_u , that we will call the *Bochner inequality*. One can further generalize the Bochner–Weitzenböck formula to a more general class of Hamiltonian systems (by dropping the positive 1-homogeneity; see [Lee, Oh4]).

Remark 2.11 (F versus $g_{\nabla u}$) In contrast to $\Delta^{\nabla u} u = \Delta u$, $\text{Ric}_N(\nabla u)$ may not coincide with the weighted Ricci curvature $\text{Ric}_N^{\nabla u}(\nabla u)$ of the weighted Riemannian manifold $(M, g_{\nabla u}, \mathbf{m})$. It is compensated in (2.10) by the fact that $\nabla^2 u$ does not necessarily coincide with the Hessian of u with respect to $g_{\nabla u}$.

The following integrated form was shown in [OS3, Theorem 3.6] for test functions $\phi \in H_c^1(M) \cap L^\infty(M)$. We will need the following slightly generalized version.

Theorem 2.12 (Integrated form) *Assume $\text{Ric}_N \geq K$ for some $K \in \mathbb{R}$ and $N \in (-\infty, 0) \cup [n, \infty]$. Given $u \in H_0^1(M) \cap H_{\text{loc}}^2(M) \cap \mathcal{C}^1(M)$ such that $\Delta u \in H_0^1(M)$, we have*

$$\begin{aligned} & - \int_M D\phi \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m} \\ & \geq \int_M \phi \left\{ D[\Delta u](\nabla u) + KF^2(\nabla u) + \frac{(\Delta u)^2}{N} \right\} d\mathbf{m} \end{aligned} \quad (2.12)$$

for all bounded nonnegative functions $\phi \in H_{\text{loc}}^1(M) \cap L^\infty(M)$.

Proof. Note that the inequality is linear in the test function ϕ . Thus one can reduce the claim to those for test functions in $H_c^1(M) \cap L^\infty(M)$ (valid by [OS1, Theorem 3.6]) by employing a partition of unity $\{h_i\}_{i \in \mathbb{N}}$ with $h_i \in \mathcal{C}_c^\infty(M)$ and decomposing ϕ as $\phi = \sum_{i=1}^\infty h_i \phi$, once we see that both sides of (2.12) are well-defined. The RHS is clearly well-defined by the conditions $u \in H_0^1(M)$, $\Delta u \in H_0^1(M)$ and $\phi \in L^\infty(M)$.

As for the LHS, observe that

$$\begin{aligned} & \sum_{i=1}^\infty \min \left\{ - \int_M D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m}, 0 \right\} \\ & \geq \sum_{i=1}^\infty \min \left\{ \int_M h_i \phi \left\{ D[\Delta u](\nabla u) + KF^2(\nabla u) + \frac{(\Delta u)^2}{N} \right\} d\mathbf{m}, 0 \right\} \\ & > -\infty. \end{aligned}$$

Replacing ϕ with $\|\phi\|_{L^\infty} - \phi$, we also find

$$\sum_{i=1}^\infty \max \left\{ - \int_M D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m}, 0 \right\} < \infty.$$

Therefore the sum

$$\sum_{i=1}^{\infty} \int_M D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m}$$

is well-defined. The independence from the choice of $\{h_i\}_{i \in \mathbb{N}}$ is seen in the standard way, that is, for another partition of unity $\{\bar{h}_j\}_{j \in \mathbb{N}}$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} \int_M D[\bar{h}_j \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m} &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int_M D[h_i \bar{h}_j \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m} \\ &= \sum_{i=1}^{\infty} \int_M D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m}. \end{aligned}$$

This completes the proof. \square

Recall from Theorem 2.8(ii) that global solutions to the heat equation always enjoy the condition $u \in H_0^1(M) \cap H_{\text{loc}}^2(M) \cap \mathcal{C}^1(M)$, and also $\Delta u \in H_0^1(M)$ when $S_F < \infty$.

Remark 2.13 (LHS of (2.12)) The LHS of (2.12) will be understood as in the proof:

$$\int_M D\phi \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m} := \sum_{i=1}^{\infty} \int_M D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m}.$$

All calculations in the following sections are done with this interpretation.

For later convenience, we introduce the following notations.

Definition 2.14 (Reverse Finsler structures) We define the *reverse Finsler structure* \overleftarrow{F} of F by $\overleftarrow{F}(v) := F(-v)$.

When we put an arrow \leftarrow on those quantities associated with \overleftarrow{F} , we have for example $\overleftarrow{d}(x, y) = d(y, x)$, $\overleftarrow{\nabla}u = -\nabla(-u)$ and $\overleftarrow{\text{Ric}}_N(v) = \text{Ric}_N(-v)$. We say that (M, F) is *backward complete* if (M, \overleftarrow{F}) is forward complete. If $\Lambda_F < \infty$, then these completenesses are mutually equivalent, and we call it simply *completeness*.

3 Linearized semigroups and gradient estimates

In the Bochner–Weitzenböck formula (Theorem 2.10) in the previous section, we used the linearized Laplacian $\Delta^{\nabla u}$ induced from the Riemannian structure $g_{\nabla u}$. In the same spirit, we can consider the linearized heat equation associated with a global solution to the heat equation. This technique turned out useful and we obtained gradient estimates à la Bakry–Émery and Li–Yau (see [OS3, §4]). In this section we discuss such a linearization in detail and improve Bakry–Émery’s L^2 -gradient estimate to an L^1 -bound (Theorem 3.7).

3.1 Linearized heat semigroups and their adjoints

Let $(u_t)_{t \geq 0}$ be a global solution to the heat equation. We will fix a measurable one-parameter family of *non-vanishing* vector fields $(V_t)_{t \geq 0}$ such that $V_t = \nabla u_t$ on M_{u_t} for each $t \geq 0$.

Given $f \in H_0^1(M)$ and $s \geq 0$, let $(P_{s,t}^{\nabla u}(f))_{t \geq s}$ be the weak solution to the *linearized heat equation*:

$$\partial_t [P_{s,t}^{\nabla u}(f)] = \Delta^{V_t} [P_{s,t}^{\nabla u}(f)], \quad P_{s,s}^{\nabla u}(f) = f. \quad (3.1)$$

The existence and other properties of the linearized semigroup $P_{s,t}^{\nabla u}$ are summarized in the following proposition (compare this with Theorem 2.8).

Proposition 3.1 (Properties of linearized semigroups) *Assume $S_F < \infty$, and let $(u_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ be as above.*

- (i) *For each $s \geq 0$, $T > 0$ and $f \in H_0^1(M)$, there exists a unique solution $f_t = P_{s,t}^{\nabla u}(f)$, $t \in [s, s+T]$, to (3.1) in the weak sense that*

$$\int_s^{s+T} \int_M \partial_t \phi_t \cdot f_t \, d\mathbf{m} \, dt = \int_s^{s+T} \int_M D\phi_t(\nabla^{V_t} f_t) \, d\mathbf{m} \, dt \quad (3.2)$$

for all $\phi \in \mathcal{C}_c^\infty((s, s+T) \times M)$.

- (ii) *The solution $(f_t)_{t \in [s, s+T]}$ is Hölder continuous on $(s, s+T) \times M$ as well as H_{loc}^2 and $\mathcal{C}^{1,\alpha}$ in x . Moreover, we have $\partial_t f_t \in H_0^1(M)$ for $t \in (s, s+T)$.*

Proof. (i) Let $s = 0$ without loss of generality. We construct the solution via a piecewise approximation. Fix $\tau > 0$ and let $(f_t^\tau)_{t \in [0, \tau]}$ be the solution to the equation

$$\partial_t f_t^\tau = \Delta^{V_0} f_t^\tau, \quad f_0^\tau = f.$$

Notice that this evolution is regarded as the gradient flow of the ‘bi-linearized’ energy form:

$$\mathcal{E}^{V_0}(h) := \frac{1}{2} \int_M g_{V_0}(\nabla^{V_0} h, \nabla^{V_0} h) \, d\mathbf{m} = \frac{1}{2} \int_M g_{\mathcal{L}(V_0)}^*(Dh, Dh) \, d\mathbf{m}.$$

In particular, the analogue of (2.8) holds true. Since

$$\partial_t (\|f_t^\tau\|_{L^2}^2) = -4\mathcal{E}^{V_0}(f_t^\tau) \leq 0, \quad (3.3)$$

we have $\|f_t^\tau\|_{L^2} \leq \|f\|_{L^2}$ for all $t \in (0, \tau]$. We recursively define $(f_t^\tau)_{t \in [k\tau, (k+1)\tau]}$ for $k \in \mathbb{N}$ as the solution to $\partial_t f_t^\tau = \Delta^{V_{k\tau}} f_t^\tau$ starting from $f_{k\tau}^\tau$ of the previous step. By construction, for any $\phi \in \mathcal{C}_c^\infty((0, T) \times M)$, we have

$$\int_0^{N\tau} \int_M \partial_t \phi_t \cdot f_t^\tau \, d\mathbf{m} \, dt = \sum_{k=0}^{N-1} \int_{k\tau}^{(k+1)\tau} \int_M D\phi_t(\nabla^{V_{k\tau}} f_t^\tau) \, d\mathbf{m} \, dt,$$

where $(N-1)\tau < T \leq N\tau$. We shall take the limit as $\tau \downarrow 0$.

By (3.3) and Lemma 2.3, we have $\|f_t^\tau\|_{L^2} \leq \|f\|_{L^2}$ for all $t > 0$ as well as the energy estimate:

$$\begin{aligned} \int_0^{N\tau} \mathcal{E}(f_t^\tau) dt &\leq S_F \sum_{k=0}^{N-1} \int_{k\tau}^{(k+1)\tau} \mathcal{E}^{V_{k\tau}}(f_t^\tau) dt = \frac{S_F}{4} \sum_{k=0}^{N-1} (\|f_{k\tau}^\tau\|_{L^2}^2 - \|f_{(k+1)\tau}^\tau\|_{L^2}^2) \\ &= \frac{S_F}{4} (\|f\|_{L^2}^2 - \|f_{N\tau}^\tau\|_{L^2}^2) \leq \frac{S_F}{4} \|f\|_{L^2}^2. \end{aligned}$$

Since $\mathcal{E}(-f_t^\tau) \leq S_F \cdot \mathcal{E}^{V_{k\tau}}(-f_t^\tau) = S_F \cdot \mathcal{E}^{V_{k\tau}}(f_t^\tau)$, we also find

$$\int_0^{N\tau} \mathcal{E}(-f_t^\tau) dt \leq \frac{S_F}{4} \|f\|_{L^2}^2.$$

Therefore a subsequence $(f_t^{\tau_i})_{t \in [0, T]}$, with $\lim_{i \rightarrow \infty} \tau_i = 0$, converges to some $(f_t)_{t \in [0, T]} \subset H_0^1(M)$ enjoying (3.2). The uniqueness follows from the non-expansion property:

$$\partial_t (\|f_t\|_{L^2}^2) = -4\mathcal{E}^{V_t}(f_t) \leq 0. \quad (3.4)$$

(ii) The Hölder continuity is a standard consequence of the local uniform ellipticity of Δ^{V_t} , see [Sal] and [OS1, Proposition 4.4]. One can show $\partial_t f_t \in H_0^1(M)$ similarly to [OS1, Appendix A]. As for the spatial derivatives, exactly the same arguments as [OS1, Appendix C] and [OS1, Theorem 4.9] yield the H_{loc}^2 - and $\mathcal{C}^{1, \alpha}$ -regularity, respectively. \square

The uniqueness in (i) above ensures that $u_t = P_{s,t}^{\nabla u}(u_s)$. It follows from (3.4) that $P_{s,t}^{\nabla u}$ uniquely extends to a contraction semigroup acting on $L^2(M)$. Notice also that

$$f \in \mathcal{C}^\infty \left(\bigcup_{s < t < s+T} (\{t\} \times M_{u_t}) \right).$$

The operator $P_{s,t}^{\nabla u}$ is linear but *nonsymmetric* (with respect to the L^2 -inner product). Let us denote by $\widehat{P}_{s,t}^{\nabla u}$ the *adjoint operator* of $P_{s,t}^{\nabla u}$. That is to say, given $\phi \in H_0^1(M)$ and $t > 0$, we define $(\widehat{P}_{s,t}^{\nabla u}(\phi))_{s \in [0, t]}$ as the solution to the equation

$$\partial_s [\widehat{P}_{s,t}^{\nabla u}(\phi)] = -\Delta^{V_s} [\widehat{P}_{s,t}^{\nabla u}(\phi)], \quad \widehat{P}_{t,t}^{\nabla u}(\phi) = \phi. \quad (3.5)$$

Note that

$$\int_M \phi \cdot P_{s,t}^{\nabla u}(f) d\mathbf{m} = \int_M \widehat{P}_{s,t}^{\nabla u}(\phi) \cdot f d\mathbf{m} \quad (3.6)$$

indeed holds since for $r \in (0, t - s)$

$$\begin{aligned} &\partial_r \left[\int_M \widehat{P}_{s+r,t}^{\nabla u}(\phi) \cdot P_{s,s+r}^{\nabla u}(f) d\mathbf{m} \right] \\ &= - \int_M \Delta^{V_{s+r}} [\widehat{P}_{s+r,t}^{\nabla u}(\phi)] \cdot P_{s,s+r}^{\nabla u}(f) d\mathbf{m} + \int_M \widehat{P}_{s+r,t}^{\nabla u}(\phi) \cdot \Delta^{V_{s+r}} [P_{s,s+r}^{\nabla u}(f)] d\mathbf{m} \\ &= 0. \end{aligned}$$

One may rewrite (3.5) as

$$\partial_\sigma[\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)] = \Delta^{V_{t-\sigma}}[\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)], \quad \sigma \in [0, t],$$

to see that the adjoint heat semigroup solves the linearized heat equation *backward in time*. (This evolution is sometimes called the *conjugate heat semigroup*, especially in the Ricci flow theory; see for instance [Ch₊, Chapter 5].) Therefore we see in the same way as $P_{s,t}^{\nabla u}$ that $\|\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)\|_{L^2}$ is non-increasing in σ and that $\hat{P}_{t-\sigma,t}^{\nabla u}$ extends to a linear contraction semigroup acting on $L^2(M)$.

Remark 3.2 In general, the semigroups $P_{s,t}^{\nabla u}$ and $\hat{P}_{s,t}^{\nabla u}$ depend on the choice of an auxiliary vector field $(V_t)_{t \geq 0}$. We will not discuss this issue, but carefully replace V_t with ∇u_t as far as it is possible.

By a well known technique based on the Bochner inequality (2.11) with $N = \infty$, we obtain the L^2 -gradient estimate of the following form.

Theorem 3.3 (L^2 -gradient estimate) *Assume $\text{Ric}_\infty \geq K$ and $S_F < \infty$. Then, given any global solution $(u_t)_{t \geq 0}$ to the heat equation with $u_0 \in \mathcal{C}_c^\infty(M)$, we have*

$$F^2(\nabla u_t(x)) \leq e^{-2K(t-s)} P_{s,t}^{\nabla u}(F^2(\nabla u_s))(x)$$

for all $0 \leq s < t < \infty$ and $x \in M$.

We remark that the condition $u_0 \in \mathcal{C}_c^\infty(M)$ ensures $F^2(\nabla u_0) \in \mathcal{C}_c^1(M)$, and hence both sides in Theorem 3.3 are Hölder continuous. By virtue of Theorem 2.12, the proof is the same as the compact case in [OS3, Theorem 4.1]. Note that we used the nonlinear semigroup $(u_s \rightarrow u_t)$ in the LHS, while in the RHS the linearized semigroup $P_{s,t}^{\nabla u}$ is employed.

Remark 3.4 In the proof of [OS3, Theorem 4.1], we did not distinguish $P_{s,t}^{\nabla u}$ and $\hat{P}_{s,t}^{\nabla u}$ and treated $P_{s,t}^{\nabla u}$ as a symmetric operator. However, the proof is valid by replacing $P_{s,t}^{\nabla u}h$ with $\hat{P}_{s,t}^{\nabla u}h$. See the proof of Theorem 3.7 below which is based on a similar calculation (with the sharper inequality in Proposition 3.5).

3.2 Improved Bochner inequality and L^1 -gradient estimate

We shall give an inequality improving the Bochner inequality (2.11) with $N = \infty$, that will be used to show the L^1 -gradient estimate as well as Bakry–Ledoux’s isoperimetric inequality. In the context of linear diffusion operators, such an inequality can be derived from (2.11) by a self-improvement argument (see [BGL, §C.6], and also [Sav] for a recent extension to $\text{RCD}(K, \infty)$ -spaces). Here we give a direct proof by calculations in coordinates.

Proposition 3.5 (Improved Bochner inequality) *Assume $\text{Ric}_\infty \geq K$ for some $K \in \mathbb{R}$. Then we have, for any $u \in \mathcal{C}^\infty(M)$,*

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) - KF^2(\nabla u) \geq D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)]) \quad (3.7)$$

point-wise on M_u .

Proof. Observe from the Bochner–Weitzenböck formula (2.10) that

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) - KF^2(\nabla u) \geq \|\nabla^2 u\|_{\text{HS}(\nabla u)}^2.$$

Therefore it suffices to show

$$4F^2(\nabla u) \|\nabla^2 u\|_{\text{HS}(\nabla u)}^2 \geq D[F^2(\nabla u)](\nabla^{\nabla u}[F^2(\nabla u)]). \quad (3.8)$$

Fix $x \in M_u$ and choose a coordinate such that $g_{ij}(\nabla u(x)) = \delta_{ij}$. We first calculate the RHS of (3.8) at x as

$$\begin{aligned} D[F^2(\nabla u)](\nabla^{\nabla u}[F^2(\nabla u)]) &= \sum_{i=1}^n \left(\frac{\partial[F^2(\nabla u)]}{\partial x^i} \right)^2 \\ &= \sum_{i=1}^n \left(\frac{\partial}{\partial x^i} \left[\sum_{j,k=1}^n g_{jk}^*(Du) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} \right] \right)^2 \\ &= \sum_{i=1}^n \left(2 \sum_{j=1}^n \frac{\partial u}{\partial x^j} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j,k=1}^n \frac{\partial g_{jk}^*}{\partial x^i}(Du) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} + \sum_{j,k,l=1}^n \frac{\partial g_{ij}^*}{\partial \alpha_l}(Du) \frac{\partial^2 u}{\partial x^i \partial x^l} \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} \right)^2 \\ &= \sum_{i=1}^n \left(2 \sum_{j=1}^n \frac{\partial u}{\partial x^j} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j,k=1}^n \frac{\partial g_{jk}^*}{\partial x^i}(Du) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} \right)^2. \end{aligned}$$

We used Euler's theorem (Theorem 2.2, similarly to (2.3)) in the last equality. Next we observe from (2.6) and (2.5) that, again at x ,

$$\begin{aligned} \nabla^2 u \left(\frac{\partial}{\partial x^j} \right) &= D_{\partial/\partial x^j}^{\nabla u}(\nabla u) \\ &= \sum_{i=1}^n \left\{ \frac{\partial}{\partial x^j} \left[\sum_{k=1}^n g_{ik}^*(Du) \frac{\partial u}{\partial x^k} \right] + \sum_{k=1}^n \Gamma_{jk}^i(\nabla u) \frac{\partial u}{\partial x^k} \right\} \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left\{ \frac{\partial^2 u}{\partial x^j \partial x^i} + \sum_{k=1}^n \frac{\partial g_{ik}^*}{\partial x^j}(Du) \frac{\partial u}{\partial x^k} + \sum_{k=1}^n \gamma_{jk}^i(\nabla u) \frac{\partial u}{\partial x^k} - \sum_{l=1}^n \frac{A_{jil}(\nabla u)}{F(\nabla u)} G^l(\nabla u) \right\} \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^n \left\{ \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{k=1}^n \left(\gamma_{jk}^i - \frac{\partial g_{ik}}{\partial x^j} \right) (\nabla u) \frac{\partial u}{\partial x^k} - \sum_{k=1}^n \frac{A_{ijk} G^k}{F}(\nabla u) \right\} \frac{\partial}{\partial x^i}. \end{aligned}$$

In the last line we used

$$\frac{\partial g_{ik}^*}{\partial x^j}(Du) = -\frac{\partial g_{ik}}{\partial x^j}(\nabla u).$$

Thus we obtain from the Cauchy–Schwarz inequality, (2.3) and (2.4) that

$$\begin{aligned}
& F^2(\nabla u) \|\nabla^2 u\|_{\text{HS}(\nabla u)}^2 \\
&= \sum_{j=1}^n \left(\frac{\partial u}{\partial x^j} \right)^2 \cdot \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{k=1}^n \left(\gamma_{jk}^i - \frac{\partial g_{ik}}{\partial x^j} \right) (\nabla u) \frac{\partial u}{\partial x^k} - \sum_{k=1}^n \frac{A_{ijk} G^k}{F} (\nabla u) \right)^2 \\
&\geq \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial u}{\partial x^j} \left\{ \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{k=1}^n \left(\gamma_{jk}^i - \frac{\partial g_{ik}}{\partial x^j} \right) (\nabla u) \frac{\partial u}{\partial x^k} - \sum_{k=1}^n \frac{A_{ijk} G^k}{F} (\nabla u) \right\} \right)^2 \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial u}{\partial x^j} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j,k=1}^n \left(\gamma_{jk}^i - \frac{\partial g_{ik}}{\partial x^j} \right) (\nabla u) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} \right)^2 \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial u}{\partial x^j} \frac{\partial u^2}{\partial x^i \partial x^j} - \frac{1}{2} \sum_{j,k=1}^n \frac{\partial g_{jk}}{\partial x^i} (\nabla u) \frac{\partial u}{\partial x^j} \frac{\partial u}{\partial x^k} \right)^2.
\end{aligned}$$

Therefore we complete the proof of (3.8) as well as (3.7). \square

Corollary 3.6 (Integrated form) *Assume $\text{Ric}_\infty \geq K$ for some $K \in \mathbb{R}$. Given $u \in H_0^1(M) \cap H_{\text{loc}}^2(M) \cap \mathcal{C}^1(M)$ such that $\Delta u \in H_0^1(M)$, we have*

$$\begin{aligned}
& - \int_M D\phi \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) d\mathbf{m} \\
& \geq \int_M \phi \left\{ D[\Delta u](\nabla u) + KF^2(\nabla u) + D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)]) \right\} d\mathbf{m}
\end{aligned}$$

for all bounded nonnegative functions $\phi \in H_{\text{loc}}^1(M) \cap L^\infty(M)$.

Proof. The case of $\phi \in H_c^1(M) \cap L^\infty(M)$ is shown in the same way as [OS3, Theorem 3.6] by noticing that $D[F(\nabla u)] = 0$ almost everywhere on $M \setminus M_u$ (see [OS3, Lemma 3.5]). As for the general case of $\phi \in H_{\text{loc}}^1(M) \cap L^\infty(M)$, along the lines of Theorem 2.12, it suffices to see that the additional term

$$\int_M \phi D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)]) d\mathbf{m}$$

is well-defined. This is seen from

$$\begin{aligned}
0 &\leq \sum_{i=1}^\infty \int_M (h_i \phi) D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)]) d\mathbf{m} \\
&\leq - \sum_{i=1}^\infty \int_M \left\{ D[h_i \phi] \left(\nabla^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] \right) + h_i \phi \{ D[\Delta u](\nabla u) + KF^2(\nabla u) \} \right\} d\mathbf{m} \\
&< \infty,
\end{aligned}$$

where $\{h_i\}_{i \in \mathbb{N}}$ is a partition of unity as in the proof of Theorem 2.12 (recall Remark 2.13 as well). \square

In a similar (but more technical) manner to the derivation of the L^2 -gradient estimate (Theorem 3.3) from the Bochner inequality (2.11), the improved Bochner inequality (3.7) yields the following.

Theorem 3.7 (L^1 -gradient estimate) *Assume $\text{Ric}_\infty \geq K$, $S_F < \infty$ and the completeness of (M, F) . Then, given any global solution $(u_t)_{t \geq 0}$ to the heat equation with $u_0 \in \mathcal{C}_c^\infty(M)$, we have*

$$F(\nabla u_t(x)) \leq e^{-K(t-s)} P_{s,t}^{\nabla u}(F(\nabla u_s))(x)$$

for all $0 \leq s < t < \infty$ and $x \in M$.

Proof. The proof closely follows [BGL, Theorem 3.2.4]. Fix arbitrary $\varepsilon > 0$ and let us consider the function

$$\xi_\sigma := \sqrt{e^{-2K\sigma} F^2(\nabla u_{t-\sigma})} + \varepsilon, \quad 0 < \sigma < t - s.$$

Note from the proof of [OS3, Theorem 4.1] that

$$\frac{\partial}{\partial \sigma} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] = -\frac{\partial}{\partial t} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] = -D[\Delta u_{t-\sigma}](\nabla u_{t-\sigma}). \quad (3.9)$$

Hence we have, on the one hand,

$$\partial_\sigma \xi_\sigma = -\frac{e^{-2K\sigma}}{\xi_\sigma} \{ K F^2(\nabla u_{t-\sigma}) + D[\Delta u_{t-\sigma}](\nabla u_{t-\sigma}) \}.$$

On the other hand,

$$\begin{aligned} \Delta^{\nabla u_{t-\sigma}} \xi_\sigma &= \text{div} \left[\frac{1}{2\xi_\sigma} \{ e^{-2K\sigma} \nabla^{\nabla u_{t-\sigma}} [F^2(\nabla u_{t-\sigma})] \} \right] \\ &= \frac{e^{-2K\sigma}}{\xi_\sigma} \Delta^{\nabla u_{t-\sigma}} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] - \frac{e^{-2K\sigma}}{\xi_\sigma^2} D\xi_\sigma \left(\nabla^{\nabla u_{t-\sigma}} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] \right) \\ &= \frac{e^{-2K\sigma}}{\xi_\sigma} \Delta^{\nabla u_{t-\sigma}} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] - \frac{e^{-4K\sigma}}{\xi_\sigma^3} D \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] \left(\nabla^{\nabla u_{t-\sigma}} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] \right) \\ &\geq \frac{e^{-2K\sigma}}{\xi_\sigma} \Delta^{\nabla u_{t-\sigma}} \left[\frac{F^2(\nabla u_{t-\sigma})}{2} \right] - \frac{e^{-2K\sigma}}{\xi_\sigma} D[F(\nabla u_{t-\sigma})](\nabla^{\nabla u_{t-\sigma}} [F(\nabla u_{t-\sigma})]). \end{aligned}$$

Therefore the improved Bochner inequality (Corollary 3.6) shows that

$$\Delta^{\nabla u_{t-\sigma}} \xi_\sigma + \partial_\sigma \xi_\sigma \geq 0 \quad (3.10)$$

in the weak sense.

For nonnegative functions $\phi, \psi \in \mathcal{C}_c^\infty(M)$ and $\sigma \in (0, t - s)$, set

$$\Phi(\sigma) := \int_M \phi \cdot P_{t-\sigma, t}^{\nabla u}(\psi \xi_\sigma) d\mathbf{m} = \int_M \widehat{P}_{t-\sigma, t}^{\nabla u}(\phi) \cdot \psi \xi_\sigma d\mathbf{m}.$$

We deduce from (3.5) and (2.9) that

$$\begin{aligned}
\Phi'(\sigma) &= \int_M \Delta^{V_{t-\sigma}} [\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)] \cdot \psi \xi_\sigma \, d\mathbf{m} + \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot \psi \partial_\sigma \xi_\sigma \, d\mathbf{m} \\
&= \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot \Delta^{V_{t-\sigma}}(\psi \xi_\sigma) \, d\mathbf{m} + \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot \psi \partial_\sigma \xi_\sigma \, d\mathbf{m} \\
&= \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \{ \xi_\sigma \Delta^{V_{t-\sigma}} \psi + \psi \Delta^{\nabla u_{t-\sigma}} \xi_\sigma + 2D\psi(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) \} \, d\mathbf{m} \\
&\quad + \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot \psi \partial_\sigma \xi_\sigma \, d\mathbf{m}.
\end{aligned}$$

Thus, by (3.10),

$$\begin{aligned}
\Phi'(\sigma) &\geq \int_M \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \{ \xi_\sigma \Delta^{V_{t-\sigma}} \psi + 2D\psi(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) \} \, d\mathbf{m} \\
&= \int_M \left\{ \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot D\psi(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) - \xi_\sigma \cdot D[\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)](\nabla^{V_{t-\sigma}} \psi) \right\} \, d\mathbf{m}.
\end{aligned}$$

Hence we find

$$\begin{aligned}
&\Phi(t-s) - \Phi(0) \\
&\geq \int_0^{t-s} \int_M \left\{ \hat{P}_{t-\sigma,t}^{\nabla u}(\phi) \cdot D\psi(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) - \xi_\sigma \cdot D[\hat{P}_{t-\sigma,t}^{\nabla u}(\phi)](\nabla^{V_{t-\sigma}} \psi) \right\} \, d\mathbf{m} \, d\sigma. \quad (3.11)
\end{aligned}$$

We are going to apply the inequality (3.11) to $\psi_k \in \mathcal{C}_c^\infty(M)$, $k \in \mathbb{N}$, with $\psi_k \uparrow 1$ monotonically and $\|F^*(D\psi_k)\|_{L^\infty} \rightarrow 0$, and pass to the limit. We remark that such a sequence $\{\psi_k\}_{k \in \mathbb{N}}$ exists due to the completeness. To this end, we observe

$$\begin{aligned}
D\xi_\sigma(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) &= \frac{e^{-4K\sigma}}{4\xi_\sigma^2} D[F^2(\nabla u_{t-\sigma})](\nabla^{\nabla u_{t-\sigma}} [F^2 \nabla u_{t-\sigma}]) \\
&\leq \frac{e^{-2K\sigma}}{4F^2(\nabla u_{t-\sigma})} D[F^2(\nabla u_{t-\sigma})](\nabla^{\nabla u_{t-\sigma}} [F^2 \nabla u_{t-\sigma}]) \\
&\leq e^{-2K\sigma} \|\nabla^2 u_{t-\sigma}\|_{\text{HS}(\nabla u_{t-\sigma})}^2,
\end{aligned}$$

where the last inequality follows from (3.8). Now, it follows from (2.10) that, for any $f \in \mathcal{C}_c^\infty(M)$,

$$\begin{aligned}
\int_M \|\nabla^2 f\|_{\text{HS}(\nabla f)}^2 \, d\mathbf{m} &\leq \int_M \left\{ \Delta^{\nabla f} \left[\frac{F^2(\nabla f)}{2} \right] - D[\Delta f](\nabla f) - KF^2(\nabla f) \right\} \, d\mathbf{m} \\
&= \int_M \{(\Delta f)^2 - KDf(\nabla f)\} \, d\mathbf{m} = \int_M \{(\Delta f)^2 + Kf\Delta f\} \, d\mathbf{m} \\
&\leq \|\Delta f\|_{L^2}^2 + |K| \cdot \|f\|_{L^2} \|\Delta f\|_{L^2}.
\end{aligned}$$

Therefore, by approximating $u_{t-\sigma}$ with $f \in \mathcal{C}_c^\infty(M)$, we obtain

$$\int_M D\xi_\sigma(\nabla^{\nabla u_{t-\sigma}} \xi_\sigma) \, d\mathbf{m} \leq e^{-2K\sigma} (\|\Delta u_{t-\sigma}\|_{L^2}^2 + |K| \cdot \|u_{t-\sigma}\|_{L^2} \|\Delta u_{t-\sigma}\|_{L^2}).$$

This estimate enables us to pass to the limit of (3.11) applied to ψ_k described above, implying

$$\int_M \phi \cdot P_{s,t}^{\nabla u}(\xi_{t-s}) d\mathbf{m} \geq \int_M \phi \cdot \xi_0 d\mathbf{m}.$$

By the arbitrariness of ϕ and ε , we have

$$e^{-K(t-s)} P_{s,t}^{\nabla u}(F(\nabla u_s)) \geq F(\nabla u_t)$$

almost everywhere. Since both sides are Hölder continuous (Proposition 3.1(ii)), this completes the proof. \square

3.3 Characterizations of lower Ricci curvature bounds

We close the section with several characterizations of the lower Ricci curvature bound $\text{Ric}_\infty \geq K$.

Theorem 3.8 (Characterizations of Ricci curvature bounds) *Suppose $S_F < \infty$, and let (M, F) be complete. Then, for each $K \in \mathbb{R}$, the following are equivalent:*

(I) $\text{Ric}_\infty \geq K$.

(II) *The Bochner inequality*

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) \geq K F^2(\nabla u)$$

holds on M_u for all $u \in C^\infty(M)$.

(III) *The improved Bochner inequality*

$$\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) - K F^2(\nabla u) \geq D[F(\nabla u)](\nabla^{\nabla u}[F(\nabla u)])$$

holds on M_u for all $u \in C^\infty(M)$.

(IV) *The L^2 -gradient estimate*

$$F^2(\nabla u_t) \leq e^{-2K(t-s)} P_{s,t}^{\nabla u}(F^2(\nabla u_s)), \quad 0 \leq s < t < \infty,$$

holds for all global solutions $(u_t)_{t \geq 0}$ to the heat equation with $u_0 \in \mathcal{C}_c^\infty(M)$.

(V) *The L^1 -gradient estimate*

$$F(\nabla u_t) \leq e^{-K(t-s)} P_{s,t}^{\nabla u}(F(\nabla u_s)), \quad 0 \leq s < t < \infty,$$

holds for all global solutions $(u_t)_{t \geq 0}$ to the heat equation with $u_0 \in \mathcal{C}_c^\infty(M)$.

Proof. We have shown (I) \Rightarrow (III) in Proposition 3.5, and (III) \Rightarrow (V) in Theorem 3.7. The implication (V) \Rightarrow (IV) is a consequence of a kind of Jensen's inequality:

$$P_{s,t}^{\nabla u}(f)^2 \leq P_{s,t}^{\nabla u}(f^2) \quad (3.12)$$

for $f \in \mathcal{C}_c^\infty(M)$. To see (3.12), for $\psi \in \mathcal{C}_c^\infty(M)$ with $0 \leq \psi \leq 1$ and $r \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq P_{s,t}^{\nabla u}((rf + \psi)^2) = r^2 P_{s,t}^{\nabla u}(f^2) + 2r P_{s,t}^{\nabla u}(f\psi) + P_{s,t}^{\nabla u}(\psi^2) \\ &\leq r^2 P_{s,t}^{\nabla u}(f^2) + 2r P_{s,t}^{\nabla u}(f\psi) + 1. \end{aligned}$$

Letting $\psi \rightarrow 1$ in $L^2(M)$, we find $r^2 P_{s,t}^{\nabla u}(f^2) + 2r P_{s,t}^{\nabla u}(f) + 1 \geq 0$ for all $r \in \mathbb{R}$. Hence $P_{s,t}^{\nabla u}(f)^2 - P_{s,t}^{\nabla u}(f^2) \leq 0$ as desired.

One can easily deduce (IV) \Rightarrow (II) from the proof of [OS3, Theorem 4.1] (both are equivalent to $H'(s) \leq 0$ in the proof). We finally prove (II) \Rightarrow (I). Given $v_0 \in T_{x_0}M \setminus 0$, fix a coordinate $(x^i)_{i=1}^n$ around x_0 with $g_{ij}(v_0) = \delta_{ij}$ and $x^i(x_0) = 0$ for all i . Consider the function

$$u := \sum_{i=1}^n v_0^i x^i + \frac{1}{2} \sum_{i,j,k=1}^n \Gamma_{ij}^k(v_0) v_0^k x^i x^j$$

on a neighborhood of x_0 , and observe that $\nabla u(x_0) = v_0$ as well as $(\nabla^2 u)|_{T_{x_0}M} = 0$ (see [OS3, Lemma 2.3] for the precise expression in coordinates of $\nabla^2 u$). Then the Bochner–Weitzenböck formula (2.10) and (II) give

$$\text{Ric}_\infty(v_0) = \left(\Delta^{\nabla u} \left[\frac{F^2(\nabla u)}{2} \right] - D[\Delta u](\nabla u) \right)(x_0) \geq K F^2(v_0).$$

This completes the proof. \square

Remark 3.9 (The lack of contraction) In the Riemannian context, lower Ricci curvature bounds are also equivalent to contraction estimates of heat flow with respect to the Wasserstein distance (we refer to [vRS] for the Riemannian case, and [EKS] for the case of RCD-spaces). More generally, for linear semigroups, gradient estimates are directly equivalent to the corresponding contraction properties (see [Ku]). In our Finsler setting, however, the lack of the *commutativity* (introduced and studied in [OP]) prevents such a contraction estimate, at least in the same form (see [OS2]).

Remark 3.10 (Similarities to (super) Ricci flow theory) The methods in this section have connections with the Ricci flow theory. The Ricci flow gives time-dependent Riemannian metrics obeying a kind of heat equation on the space of Riemannian metrics, while we considered the time-dependent (singular) Riemannian structures $g_{\nabla u}$ for u solving the heat equation. More precisely, what corresponds to our lower Ricci curvature bound is the *super Ricci flow* (super-solutions to the Ricci flow equation). We refer to [MT] for an inspiring work on a characterization of the super Ricci flow in terms of the contraction of heat flow, and to [St3] for a recent investigation of the super Ricci flow on time-dependent metric measure spaces including various characterizations related to Theorem 3.8. Then, again, what is missing in our Finsler setting is the contraction property. As we noted in Remark 3.9 above, the Riemannian nature of the space is known to be necessary for contraction estimates.

4 Bakry–Ledoux’s isoperimetric inequality

This section is devoted to a geometric application of the improved Bochner inequality. We will assume $\text{Ric}_\infty \geq K > 0$, then $\mathfrak{m}(M) < \infty$ holds (see [St1, Theorem 4.26]) and hence we can normalize \mathfrak{m} as $\mathfrak{m}(M) = 1$ without changing Ric_∞ (\mathfrak{m} and $c\mathfrak{m}$ have the same weighted Ricci curvatures for any $c > 0$).

For a Borel set $A \subset M$, define the *Minkowski exterior boundary measure* as

$$\mathfrak{m}^+(A) := \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(B^+(A, \varepsilon)) - \mathfrak{m}(A)}{\varepsilon},$$

where $B^+(A, \varepsilon) := \{y \in M \mid \inf_{x \in A} d(x, y) < \varepsilon\}$ is the forward ε -neighborhood of A . Then the (forward) *isoperimetric profile* $\mathcal{I}_{(M, F, \mathfrak{m})} : [0, 1] \rightarrow [0, \infty)$ of (M, F, \mathfrak{m}) is defined by

$$\mathcal{I}_{(M, F, \mathfrak{m})}(\theta) := \inf\{\mathfrak{m}^+(A) \mid A \subset M : \text{Borel set with } \mathfrak{m}(A) = \theta\}.$$

Clearly $\mathcal{I}_{(M, F, \mathfrak{m})}(0) = \mathcal{I}_{(M, F, \mathfrak{m})}(1) = 0$. The main theorem in this section is the following.

Theorem 4.1 (Bakry–Ledoux’s isoperimetric inequality) *Assume that (M, F) is complete and satisfies $\text{Ric}_\infty \geq K > 0$, $\mathfrak{m}(M) = 1$ and $S_F < \infty$. Then we have*

$$\mathcal{I}_{(M, F, \mathfrak{m})}(\theta) \geq \mathcal{I}_K(\theta) \tag{4.1}$$

for all $\theta \in [0, 1]$, where

$$\mathcal{I}_K(\theta) := \sqrt{\frac{K}{2\pi}} e^{-Kc^2(\theta)/2} \quad \text{with } \theta = \int_{-\infty}^{c(\theta)} \sqrt{\frac{K}{2\pi}} e^{-Ka^2/2} da.$$

Recall that, under $S_F < \infty$, the forward completeness is equivalent to the backward completeness. In the Riemannian case, the inequality (4.1) is due to Bakry and Ledoux [BL] and can be regarded as the dimension-free version of *Lévy–Gromov’s isoperimetric inequality* (see [Lé1, Lé2, Gr]). Lévy–Gromov’s classical isoperimetric inequality states that the isoperimetric profile of an n -dimensional Riemannian manifold (M, g) with $\text{Ric} \geq n - 1$ is bounded below by the profile of the unit sphere \mathbb{S}^n (both spaces are equipped with the normalized volume measures). In (4.1), the role of the unit sphere is played by the real line \mathbb{R} equipped with the Gaussian measure $\sqrt{K/2\pi} e^{-Kx^2/2} dx$, thus (4.1) is also called the *Gaussian isoperimetric inequality*.

In [Oh7], generalizing Cavalletti and Mondino’s localization technique in [CM] inspired by Klartag’s work [Kl] on Riemannian manifolds, we showed the following slightly weaker inequality (recall the introduction for a more precise account):

$$\mathcal{I}_{(M, F, \mathfrak{m})}(\theta) \geq \Lambda_F^{-1} \cdot \mathcal{I}_K(\theta), \quad \Lambda_F = \sup_{v \in TM \setminus 0} \frac{F(v)}{F(-v)},$$

under the finite reversibility $\Lambda_F < \infty$ (but without $S_F < \infty$). In fact we have treated in [Oh7] the general curvature-dimension-diameter bound $\text{Ric}_N \geq K$ and $\text{diam } M \leq D$ (in accordance with [Mil]). Theorem 4.1 sharpens the estimate in [Oh7] in the special case of $N = D = \infty$ and $K > 0$.

4.1 Ergodicity

We begin with auxiliary properties induced from our hypothesis $\text{Ric}_\infty \geq K > 0$.

Lemma 4.2 (Global Poincaré inequality) *Suppose that (M, F) is forward or backward complete, $\text{Ric}_\infty \geq K > 0$ and $\mathbf{m}(M) = 1$. Then we have, for any locally Lipschitz function $f \in H_0^1(M)$,*

$$\int_M f^2 d\mathbf{m} - \left(\int_M f d\mathbf{m} \right)^2 \leq \frac{1}{K} \int_M F^*(Df)^2 d\mathbf{m}. \quad (4.2)$$

Proof. It is well known that the curvature bound $\text{Ric}_\infty \geq K$ (or $\text{CD}(K, \infty)$) implies the log-Sobolev inequality:

$$\int_M \rho \log \rho d\mathbf{m} \leq \frac{1}{2K} \int_M \frac{F^*(D\rho)^2}{\rho} d\mathbf{m} \quad (4.3)$$

for nonnegative locally Lipschitz functions ρ with $\int_M \rho d\mathbf{m} = 1$, and that (4.2) follows from (4.3) (see [OV, LV, Vi, Oh2]). Here we only explain the latter step taking care of the non-compactness of M .

Let us first assume that f is bounded. Since

$$\int_M f^2 d\mathbf{m} - \left(\int_M f d\mathbf{m} \right)^2 = \int_M \left(f - \int_M f d\mathbf{m} \right)^2 d\mathbf{m},$$

we can further assume that $\int_M f d\mathbf{m} = 0$. There is nothing to prove if $f \equiv 0$, thus assume $\|f\|_{L^\infty} > 0$. For $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < \|f\|_{L^\infty}^{-1}$, we consider $\mu_\varepsilon := (1 + \varepsilon f)\mathbf{m}$. Then the log-Sobolev inequality under $\text{Ric}_\infty \geq K$ implies

$$\int_M (1 + \varepsilon f) \log(1 + \varepsilon f) d\mathbf{m} \leq \frac{1}{2K} \int_M \frac{\varepsilon^2 F^*(Df)^2}{1 + \varepsilon f} d\mathbf{m}.$$

Expanding the LHS at $\varepsilon = 0$ gives

$$\int_M \left\{ \varepsilon f + \frac{1}{2}(\varepsilon f)^2 + O(\varepsilon^3) \right\} d\mathbf{m} = \frac{\varepsilon^2}{2} \int_M f^2 d\mathbf{m} + O(\varepsilon^3),$$

where $O(\varepsilon^3)$ in the LHS is uniform in M thanks to the boundedness of f . Hence we have

$$\frac{\varepsilon^2}{2} \int_M f^2 d\mathbf{m} \leq \frac{1}{1 - \varepsilon\|f\|_{L^\infty}} \frac{\varepsilon^2}{2K} \int_M F^*(Df)^2 d\mathbf{m} + O(\varepsilon^3).$$

Dividing both sides by ε^2 and letting $\varepsilon \rightarrow 0$ yields (4.2).

When f is unbounded, we consider the truncations $f_k := \min\{\max\{f, -k\}, k\}$ for $k \in \mathbb{N}$. Apply (4.2) to f_k to see

$$\int_M f_k^2 d\mathbf{m} - \left(\int_M f_k d\mathbf{m} \right)^2 \leq \frac{1}{K} \int_M F^*(Df_k)^2 d\mathbf{m} \leq \frac{1}{K} \int_M F^*(Df)^2 d\mathbf{m}.$$

Letting $k \rightarrow \infty$, we obtain (4.2) for f . □

The LHS of (4.2) is the *variance* of f :

$$\mathrm{Var}_{\mathbf{m}}(f) := \int_M f^2 d\mathbf{m} - \left(\int_M f d\mathbf{m} \right)^2.$$

We next show that the Poincaré inequality (4.2) yields the exponential decay of the variance and a kind of *ergodicity* (similarly to [BGL, §4.2]), which is one of the key ingredients in the proof of Theorem 4.1 (see the proof of Corollary 4.5). Given a global solution $(u_t)_{t \geq 0}$ to the heat equation, since the finiteness of the total mass ($\mathbf{m}(M) = 1$) together with the forward and backward completeness implies $1 \in H_0^1(M)$, we observe the mass conservation:

$$\int_M P_{s,t}^{\nabla u}(f) d\mathbf{m} = \int_M f d\mathbf{m} \quad (4.4)$$

for any $f \in H_0^1(M)$ and $0 \leq s < t < \infty$.

Proposition 4.3 (Variance decay and ergodicity) *Assume that (M, F) is forward and backward complete, $\mathrm{Ric}_\infty \geq K > 0$ and $\mathbf{m}(M) = 1$. Then we have, given any global solution $(u_t)_{t \geq 0}$ to the heat equation and $f \in H_0^1(M)$,*

$$\mathrm{Var}_{\mathbf{m}}(P_{s,t}^{\nabla u}(f)) \leq e^{-2K(t-s)/S_F} \mathrm{Var}_{\mathbf{m}}(f)$$

for all $0 \leq s < t < \infty$ (when $S_F = \infty$ the claim is read as $\mathrm{Var}_{\mathbf{m}}(P_{s,t}^{\nabla u}(f)) \leq \mathrm{Var}_{\mathbf{m}}(f)$).

In particular, if $S_F < \infty$, then $P_{s,t}^{\nabla u}(f)$ converges to the constant function $\int_M f d\mathbf{m}$ in $L^2(M)$ as $t \rightarrow \infty$.

Proof. Put $f_t := P_{s,t}^{\nabla u}(f)$, then $\int_M f_t d\mathbf{m} = \int_M f d\mathbf{m}$ holds by (4.4). It follows from Lemmas 2.3, 4.2 that

$$\begin{aligned} \frac{d}{dt}[\mathrm{Var}_{\mathbf{m}}(f_t)] &= -2 \int_M Df_t(\nabla^{V_t} f_t) d\mathbf{m} = -2 \int_M g_{\mathcal{L}(V_t)}^*(Df_t, Df_t) d\mathbf{m} \\ &\leq -\frac{2}{S_F} \int_M F^*(Df_t)^2 d\mathbf{m} \leq -\frac{2K}{S_F} \mathrm{Var}_{\mathbf{m}}(f_t). \end{aligned}$$

Thus $e^{2Kt/S_F} \mathrm{Var}_{\mathbf{m}}(f_t)$ is non-increasing in t and we complete the proof of the first assertion. The second assertion is straightforward since

$$\mathrm{Var}_{\mathbf{m}}(f_t) = \int_M \left(f_t - \int_M f d\mathbf{m} \right)^2 d\mathbf{m} \rightarrow 0 \quad (t \rightarrow \infty).$$

□

4.2 Key estimate

We next prove a key estimate which would have further applications (see [BL]). Define

$$\varphi(c) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^c e^{-b^2/2} db \quad \text{for } c \in \mathbb{R}, \quad \mathcal{N}(\theta) := \varphi' \circ \varphi^{-1}(\theta) \quad \text{for } \theta \in (0, 1).$$

Set also $\mathcal{N}(0) = \mathcal{N}(1) := 0$. We will use the relation $\mathcal{N}'' = -1/\mathcal{N}$ on $(0, 1)$.

Theorem 4.4 Assume $\text{Ric}_\infty \geq K$ for some $K \in \mathbb{R}$ and $S_F < \infty$. Then we have, given a global solution $(u_t)_{t \geq 0}$ to the heat equation with $u_0 \in \mathcal{C}_c^\infty(M)$ and $0 \leq u_0 \leq 1$,

$$\sqrt{\mathcal{N}^2(u_t) + \alpha F^2(\nabla u_t)} \leq P_{0,t}^{\nabla u} \left(\sqrt{\mathcal{N}^2(u_0) + c_\alpha(t) F^2(\nabla u_0)} \right) \quad (4.5)$$

for all $\alpha \geq 0$ and $t > 0$, where

$$c_\alpha(t) := \frac{1 - e^{-2Kt}}{K} + \alpha e^{-2Kt} > 0$$

and $c_\alpha(t) := 2t + \alpha$ when $K = 0$.

For simplicity, we suppressed the dependence of c_α on K .

Proof. Recall from (2.8) that $0 \leq u_0 \leq 1$ implies $0 \leq u_t \leq 1$ for all $t > 0$, and hence $\mathcal{N}(u_t)$ makes sense. Fix $t > 0$ and put

$$\zeta_s := \sqrt{\mathcal{N}^2(u_s) + c_\alpha(t-s) F^2(\nabla u_s)}, \quad 0 \leq s \leq t$$

(compare this function with ξ_σ in the proof of Theorem 3.7). Then (4.5) is written as $\zeta_t \leq P_{0,t}^{\nabla u}(\zeta_0)$ and our goal is to show $\partial_s[P_{s,t}^{\nabla u}(\zeta_s)] \leq 0$. Observe from (3.6) and (3.5) that, for any $\phi \in \mathcal{C}_c^\infty((0, t) \times M)$,

$$\begin{aligned} \int_0^t \int_M \phi_s \cdot \partial_s[P_{s,t}^{\nabla u}(\zeta_s)] \, d\mathbf{m} \, ds &= - \int_0^t \int_M \partial_s \phi_s \cdot P_{s,t}^{\nabla u}(\zeta_s) \, d\mathbf{m} \, ds \\ &= - \int_0^t \int_M \widehat{P}_{s,t}^{\nabla u}(\partial_s \phi_s) \cdot \zeta_s \, d\mathbf{m} \, ds \\ &= - \int_0^t \int_M \{ \partial_s[\widehat{P}_{s,t}^{\nabla u}(\phi_s)] + \Delta^{V_s}[\widehat{P}_{s,t}^{\nabla u}(\phi_s)] \} \cdot \zeta_s \, d\mathbf{m} \, ds \\ &= \int_0^t \int_M \widehat{P}_{s,t}^{\nabla u}(\phi_s) \cdot (\partial_s \zeta_s - \Delta^{\nabla u_s} \zeta_s) \, d\mathbf{m} \, ds \\ &= \int_0^t \int_M \phi_s \cdot P_{s,t}^{\nabla u}(\partial_s \zeta_s - \Delta^{\nabla u_s} \zeta_s) \, d\mathbf{m} \, ds. \end{aligned}$$

Therefore

$$\partial_s[P_{s,t}^{\nabla u}(\zeta_s)] = P_{s,t}^{\nabla u}(\partial_s \zeta_s - \Delta^{\nabla u_s} \zeta_s)$$

and it is sufficient to prove $\Delta^{\nabla u_s} \zeta_s - \partial_s \zeta_s \geq 0$ for $0 < s < t$.

On the closed set $u_s^{-1}(0) \cup u_s^{-1}(1)$, we have $Du_s = \partial_s u_s \equiv 0$ and hence $\Delta^{\nabla u_s} \zeta_s = \partial_s \zeta_s = 0$ almost everywhere. On $M \setminus (u_s^{-1}(0) \cup u_s^{-1}(1))$, we first calculate by using (3.9) and $c'_\alpha = 2(1 - Kc_\alpha)$ as

$$\partial_s \zeta_s = \frac{1}{\zeta_s} \left\{ \mathcal{N}(u_s) \mathcal{N}'(u_s) \Delta u_s + (Kc_\alpha(t-s) - 1) F^2(\nabla u_s) + c_\alpha(t-s) D[\Delta u_s](\nabla u_s) \right\}.$$

Next, we have

$$\nabla^{\nabla u_s} \zeta_s = \frac{1}{\zeta_s} \left\{ \mathcal{N}(u_s) \mathcal{N}'(u_s) \nabla u_s + \frac{c_\alpha(t-s)}{2} \nabla^{\nabla u_s} [F^2(\nabla u_s)] \right\}.$$

Hence

$$\begin{aligned}
\Delta^{\nabla u_s} \zeta_s &= \frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s} \Delta u_s + D \left[\frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s} \right] (\nabla u_s) \\
&\quad + \frac{c_\alpha(t-s)}{2\zeta_s} \Delta^{\nabla u_s} [F^2(\nabla u_s)] - \frac{c_\alpha(t-s)}{2\zeta_s^2} D\zeta_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]) \\
&= \frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s} \Delta u_s + \frac{\mathcal{N}'(u_s)^2 - 1}{\zeta_s} F^2(\nabla u_s) - \frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s^2} D\zeta_s (\nabla u_s) \\
&\quad + \frac{c_\alpha(t-s)}{2\zeta_s} \Delta^{\nabla u_s} [F^2(\nabla u_s)] - \frac{c_\alpha(t-s)}{2\zeta_s^2} D\zeta_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]),
\end{aligned}$$

where we used $\mathcal{N}'' = -1/\mathcal{N}$. (Precisely, the term $\Delta^{\nabla u_s} [F^2(\nabla u_s)]$ is understood in the weak sense.) Now we apply the improved Bochner inequality (Corollary 3.6) to obtain

$$\begin{aligned}
\Delta^{\nabla u_s} \zeta_s - \partial_s \zeta_s &= \frac{\mathcal{N}'(u_s)^2 - Kc_\alpha(t-s)}{\zeta_s} F^2(\nabla u_s) \\
&\quad + \frac{c_\alpha(t-s)}{\zeta_s} \left\{ \Delta^{\nabla u_s} \left[\frac{F^2(\nabla u_s)}{2} \right] - D[\Delta u_s] (\nabla u_s) \right\} \\
&\quad - \frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s^2} D\zeta_s (\nabla u_s) - \frac{c_\alpha(t-s)}{2\zeta_s^2} D\zeta_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]) \\
&\geq \frac{\mathcal{N}'(u_s)^2}{\zeta_s} F^2(\nabla u_s) + \frac{c_\alpha(t-s)}{\zeta_s F^2(\nabla u_s)} D \left[\frac{F^2(\nabla u_s)}{2} \right] \left(\nabla^{\nabla u_s} \left[\frac{F^2(\nabla u_s)}{2} \right] \right) \\
&\quad - \frac{\mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s^2} D\zeta_s (\nabla u_s) - \frac{c_\alpha(t-s)}{2\zeta_s^2} D\zeta_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]).
\end{aligned}$$

Substituting

$$D\zeta_s = \frac{1}{\zeta_s} \left\{ \mathcal{N}(u_s) \mathcal{N}'(u_s) Du_s + \frac{c_\alpha(t-s)}{2} D[F^2(\nabla u_s)] \right\}$$

and recalling (2.9), we obtain

$$\begin{aligned}
\Delta^{\nabla u_s} \zeta_s - \partial_s \zeta_s &\geq \frac{\zeta_s^2 \mathcal{N}'(u_s)^2 - \mathcal{N}^2(u_s) \mathcal{N}'(u_s)^2}{\zeta_s^3} F^2(\nabla u_s) \\
&\quad - \frac{c_\alpha(t-s) \mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s^3} Du_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]) \\
&\quad + \frac{c_\alpha(t-s)}{\zeta_s^3} \left\{ \frac{\zeta_s^2}{F^2(\nabla u_s)} - c_\alpha(t-s) \right\} D \left[\frac{F^2(\nabla u_s)}{2} \right] \left(\nabla^{\nabla u_s} \left[\frac{F^2(\nabla u_s)}{2} \right] \right) \\
&= \frac{c_\alpha(t-s) \mathcal{N}'(u_s)^2}{\zeta_s^3} F^4(\nabla u_s) \\
&\quad - \frac{c_\alpha(t-s) \mathcal{N}(u_s) \mathcal{N}'(u_s)}{\zeta_s^3} Du_s (\nabla^{\nabla u_s} [F^2(\nabla u_s)]) \\
&\quad + \frac{c_\alpha(t-s)}{\zeta_s^3} \frac{\mathcal{N}^2(u_s)}{F^2(\nabla u_s)} D \left[\frac{F^2(\nabla u_s)}{2} \right] \left(\nabla^{\nabla u_s} \left[\frac{F^2(\nabla u_s)}{2} \right] \right).
\end{aligned}$$

Since the Cauchy–Schwarz inequality for $g_{\nabla u_s}$ yields

$$Du_s(\nabla^{\nabla u_s}[F^2(\nabla u_s)]) \leq F(\nabla u_s) \sqrt{D[F^2(\nabla u_s)](\nabla^{\nabla u_s}[F^2(\nabla u_s)])},$$

we conclude that

$$\begin{aligned} & \Delta^{\nabla u_s} \zeta_s - \partial_s \zeta_s \\ & \geq \frac{c_\alpha(t-s)\mathcal{N}'(u_s)^2}{\zeta_s^3} F^4(\nabla u_s) \\ & \quad - \frac{c_\alpha(t-s)\mathcal{N}(u_s)|\mathcal{N}'(u_s)|}{\zeta_s^3} F(\nabla u_s) \sqrt{D[F^2(\nabla u_s)](\nabla^{\nabla u_s}[F^2(\nabla u_s)])} \\ & \quad + \frac{c_\alpha(t-s)}{\zeta_s^3} \frac{\mathcal{N}^2(u_s)}{F^2(\nabla u_s)} D\left[\frac{F^2(\nabla u_s)}{2}\right] \left(\nabla^{\nabla u_s}\left[\frac{F^2(\nabla u_s)}{2}\right]\right) \\ & = \frac{c_\alpha(t-s)}{\zeta_s^3} \left(|\mathcal{N}'(u_s)| F^2(\nabla u_s) - \frac{\mathcal{N}(u_s)}{2F(\nabla u_s)} \sqrt{D[F^2(\nabla u_s)](\nabla^{\nabla u_s}[F^2(\nabla u_s)])} \right)^2 \\ & \geq 0. \end{aligned}$$

This completes the proof. \square

If $K > 0$, choosing $\alpha = K^{-1}$ and letting $t \rightarrow \infty$ in (4.5) yields the following.

Corollary 4.5 *Assume that (M, F) is complete and satisfies $\text{Ric}_\infty \geq K > 0$, $\mathbf{S}_F < \infty$ and $\mathbf{m}(M) = 1$. Then we have, for any $u \in \mathcal{C}_c^\infty(M)$ with $0 \leq u \leq 1$,*

$$\sqrt{K} \mathcal{N} \left(\int_M u \, d\mathbf{m} \right) \leq \int_M \sqrt{K \mathcal{N}^2(u) + F^2(\nabla u)} \, d\mathbf{m}. \quad (4.6)$$

Proof. Let $(u_t)_{t \geq 0}$ be the global solution to the heat equation with $u_0 = u$. Taking $\alpha = K^{-1}$, we find $c_\alpha \equiv K^{-1}$ and hence by (4.5)

$$\sqrt{K \mathcal{N}^2(u_t)} \leq \sqrt{K \mathcal{N}^2(u_t) + F^2(\nabla u_t)} \leq P_{0,t}^{\nabla u} \left(\sqrt{K \mathcal{N}^2(u) + F^2(\nabla u)} \right).$$

Letting $t \rightarrow \infty$, we deduce from the ergodicity (Proposition 4.3) that

$$\begin{aligned} u_t & \rightarrow \int_M u \, d\mathbf{m}, \\ P_{0,t}^{\nabla u} \left(\sqrt{K \mathcal{N}^2(u) + F^2(\nabla u)} \right) & \rightarrow \int_M \sqrt{K \mathcal{N}^2(u) + F^2(\nabla u)} \, d\mathbf{m} \end{aligned}$$

in $L^2(M)$. Thus we obtain (4.6). \square

4.3 Proof of Theorem 4.1

Proof. Let $\theta \in (0, 1)$. Fix a closed set $A \subset M$ with $\mathbf{m}(A) = \theta$ and consider

$$u^\varepsilon(x) := \max\{1 - \varepsilon^{-1}d(x, A), 0\}, \quad \varepsilon > 0.$$

Notice that $F(\nabla u^\varepsilon) = \varepsilon^{-1}$ on $B^-(A, \varepsilon) \setminus A$, where

$$B^-(A, \varepsilon) := \left\{ x \in M \mid \inf_{y \in A} d(x, y) < \varepsilon \right\}.$$

Applying (4.6) to (smooth approximations of) u^ε and letting $\varepsilon \downarrow 0$ implies, with the help of $\mathcal{N}(0) = \mathcal{N}(1) = 0$,

$$\sqrt{K} \mathcal{N}(\theta) \leq \liminf_{\varepsilon \downarrow 0} \frac{\mathbf{m}(B^-(A, \varepsilon)) - \mathbf{m}(A)}{\varepsilon}.$$

This is the desired isoperimetric inequality for the reverse Finsler structure \overleftarrow{F} (recall Definition 2.14), since with $c := \varphi^{-1}(\theta)/\sqrt{K}$

$$\sqrt{K} \mathcal{N}(\theta) = \sqrt{\frac{K}{2\pi}} e^{-Kc^2/2}, \quad \theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{K}c} e^{-b^2/2} db = \sqrt{\frac{K}{2\pi}} \int_{-\infty}^c e^{-Ka^2/2} da.$$

Because the curvature bound $\text{Ric}_\infty \geq K$ is common to F and \overleftarrow{F} , we also obtain (4.1). \square

The same argument as [Oh7, Corollary 7.5] gives the following corollary concerning normed spaces. Even this simple case seems new.

Corollary 4.6 (Isoperimetric inequality on normed spaces) *Let $n \geq 2$ and $\|\cdot\| : \mathbb{R}^n \rightarrow [0, \infty)$ be a continuous function satisfying:*

- (1) $\|x\| > 0$ for all $x \in \mathbb{R}^n \setminus \{(0, 0, \dots, 0)\}$;
- (2) $\|cx\| = c\|x\|$ for all $x \in \mathbb{R}^n$ and $c > 0$;
- (3) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \mathbb{R}^n$.

Consider the distance function $d(x, y) := \|y - x\|$ of \mathbb{R}^n , and take a probability measure $d\mathbf{m} = e^{-\Phi} dx^1 dx^2 \cdots dx^n$ on \mathbb{R}^n such that $dx^1 dx^2 \cdots dx^n$ is the Lebesgue measure and Φ is a continuous function.

If Φ is K -convex with $K > 0$ in the sense that

$$\Phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y) - \frac{K}{2}(1 - \lambda)\lambda d^2(x, y)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, then we have

$$\mathcal{I}_{(\mathbb{R}^n, d, \mathbf{m})}(\theta) \geq \mathcal{I}_K(\theta) \quad \text{for all } \theta \in (0, 1).$$

We remark that the completeness is clear in this case, and $S_F < \infty$ is enjoyed for smooth approximations of the norm $\|\cdot\|$.

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